Laboratory experiment and evolutionary competition in lowest unique integer games

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Abstract. This study computationally examines whether keeping on choosing the same number is really successful in small-sized lowest unique integer games (LUIGs). In a LUIG, N (≥ 3) players submit a positive integer up to M and the player choosing the smallest number not chosen by anyone else wins. For this purpose, the author considers four LUIGs with $N = \{3, 4\}$ and $M = \{3, 4\}$ and uses the behavioral data obtained from the laboratory experiment by Yamada and Hanaki (Physica A, *in press*). For computational experiments, the author estimates the parameters of typical learning models for each subject and then pursues evolutionary competitions. The main but preliminary findings are in the following: First, sticking behaviors such as Level-1 and Level-2 computationally perform well as well as experimentally. Second, as the generations go by, the frequency of average changes decreases while the game efficiency differs from the game setup, namely improved in three-person LUIGs but not in four-person LUIGs.

1 Introduction

In social and economic systems, individuals, groups, firms and so on make their decision based on the rules they should follow. For example, call market, continuous double auction and other trading mechanisms are seen in financial markets. Or, first- and second-prize styles are usually employed in auction markets. On the other hand, new types of social and economic systems have been proposed and some of them are introduced in practice. Among these, Swedish lottery (SL) game Limbo and Lowest/Highest Unique Bid Auctions (LUBA /HUBA) like the Auction Air or Juubeo websites are one of the new systems where the participants are required to be unique by taking risks of not being so.

Lowest Unique Integer Games (LUIGs) are highly simplified versions of real systems. In a LUIG, N (≥ 3) players simultaneously submit a positive integer up to M. The player choosing the smallest number that is not chosen by anyone else i s the winner. In cases where no player chooses a unique number, there is no winner. For instance, suppose there is a LUIG with $N = 3$ and $M = 3$. There are three players, A, B, and C, who each submit an integer between 1 and 3. If the integer s chosen by A, B, and C are 1, 2, and 3, respectively, then A wins the game. If the integers chosen by A, B and C are 1, 1, and 2, respectively, then C is the w inner. And, as noted, if all of them choose the same integer, there is no winner.

Hence, LUIGs are more tractable than the above real systems because the exact numbers of players or participants and the options are known for their decision-making. In addition, unlike with LUIGs or SL, in LUBA/HUBA scenarios, a winner has to pay the amount s/he bids in exchange for the item being auctioned. In this sense, these types of real systems have been attracting much attention recently from scholars of various disciplines [3, 4, 6, 8, 9, 11, 12, 14–17, 19]. While the studies mentioned investigate these related systems theoretically and empirically, experimental studies on LUIGs and related systems are still scarce except for Östling et al. [13] and Ohtsubo et al. [14]. Yamada and Hanaki experimentally study LUIGs to determine if and how subjects self-organize into different behavioral classes to obtain insights into choice patterns that can shed light on the alleviation of congestion problems [18]. They consider four LUIGs with $N = \{3, 4\}$ and $M = \{3, 4\}$. They find that (a) choices made by more than 1/3 of subjects were not significantly different from what a symmetric mixed-strategy Nash equilibrium (MSE) predicts; however, (b) subjects who behaved significantly differently from what the MSE predicts won the game more frequently. What distinguishes subjects was their tendencies to change their choices following losses.

This study extends their past experimental study to check whether such successful or unsuccessful behaviors are always observed no matter whom their opponents are. For this purpose, the author pursues computational approach in line with the work by Linde et al. [10] who carried out strategy experiment for Minority Game (MG) of five and then evolutionary competition using the submitted strategies. Here, several typical learning and strategic thinking models are employed to express the behaviors of subjects obtained in the laboratory experiment. Then, the one with the best likelihood for every subject in each game setup is used for computational experiments. Finally, by making comparison between the work by Linde et al. and this one, the author discusses the structural and behavioral differences between MGs and LUIGs because both the two games are sometimes considered as similar.

The remainder of this paper is organized as follows: The next section defines lowest unique integer game. Section 3 summarizes the laboratory experiment by Yamada and Hanaki with the main results. Section 4 explains the learning models used for computational experiments and then presents a couple of preliminary results. Section 5 concludes.

2 Lowest unique integer game

There are N players who each choose one positive integer from 1 to M (> 1). All of them know this setup. The player who submits the smallest integer that is not chosen by anyone else is a winner. The winner receives a positive payoff, usually normalized 1, and the losers do zero. If there is no uniquely chosen integer, all players become losers.

Table 1. Symmetric mixed strategy equilibrium in LUIG

N	M	1	2	3	4
3	3	0.464	0.268	0.268	
3	4	0.458	0.252	0.145	0.145
4	3	0.449	0.426	0.125	
4	4	0.448	0.425	0.126	0.002

Table 2. Two games played in each session

Session	Game 1	Game 2
1	$(N, M) = (3, 3)$	$(N, M) = (3, 4)$
2	$(N, M) = (3, 4)$	$(N, M) = (4, 4)$
3	$(N, M) = (4, 4)$	$(N, M) = (4, 3)$
4	$(N, M) = (4, 3)$	$(N, M) = (3, 3)$
5	$(N, M) = (3, 4)$	$(N, M) = (3, 3)$
6	$(N, M) = (3, 3)$	$(N, M) = (4, 3)$
7	$(N, M) = (4, 3)$	$(N, M) = (4, 4)$
8	$(N, M) = (4, 4)$	$(N, M) = (3, 4)$

Here the author considers LUIG with $N \geq 3$ and $M \geq 3$. In case of bi-matrix game, there are three equilibria, (1) both players choose 1 and (2) one player chooses 1 and the other does 2. But, since one never makes one's opponent a winner so long as s/he keeps on choosing 1 [13], this kind of game is not worth investigating.

Then, each game form has a unique mixed strategy equilibrium. Table 1 gives the mixed strategy equilibria in cases of $(N, M) = (3, 3),\ (3, 4),\ (4, 3),$ and $(4, 4)$.[1]

3 Laboratory experiment

Yamada and Hanaki considered four LUIGs with $N \in \{3, 4\}$ and $M \in \{3, 4\}$ [18]. Each subject played two separate LUIGs. They changed either N or M, but not both, between the two games a subject played. Thus, there were totally eight pairs of games as shown in Table 2. Each LUIG had 50 rounds with the same group of subjects. There was a non-binding time limit of 15 seconds for choosing an integer in each round. After every subject in the group made his/her choice, the subjects were informed of the result of the round. The feedback consisted of whether a subject was a winner or not, in addition to the winning number for the round. Subjects were informed that the winning number was set to zero when there was no winner.

Once the 50 rounds of the first LUIG were completed, subjects were rematched to form another group to play the second LUIG for 50 rounds. Subjects

[1] Östling et al. have a succinct algorithm to calculate mixed strategy equilibrium in this setup [13].

were initially told that they would play two LUIGs with 50 rounds each, but were not informed about the exact game (i.e., N and M) until the start of each game. At the beginning of each game, they were reminded that the other subjects in their group would remain the same during the 50 rounds.

Subjects were paid according to the outcome of one randomly chosen round from each game. The winner of a game received 20 euros in addition to a participation fee of 10 euros. Thus a subject could earn a maximum of 50 euros. Subjects were paid in cash at the end of the experiment.[2]

Computerized experiments, implemented using z-Tree [7], took place in January and February 2014 at the Laboratoire d'Expérimentation en Sciences Sociales et Analyse des Comportements (LESSAC), Burgundy School of Business (Dijon, France). 192 students who had never experienced a LUIG experiment participated. There were 24 students in each of the 8 sessions as listed in Table 2. A session lasted between 65 and 85 minutes. Out of our 192 subjects, 11 earned 50 euros and 67 earned 30 euros. The remaining 114 subjects earned only the participation fee of 10 euros.

Table 3 shows the relative frequencies of the observed winning numbers in the four LUIGs. The authors are pooling all the groups that played the relevant LUIG. For each LUIG, the predicted relative frequencies under the MSE are also reported. Recall that a winning number "0" represents a case without any winner. For three out of four LUIGs, namely, LUIG33, LUIG43, and LUIG44, the observed frequencies of winning numbers are very similar to what the MSE of each game predicts. The major difference between the prediction of the MSE and the experimental outcome is observed in LUIG34 in which "4" was much less frequently the winning number in the experiment compared to the MSE.

Table 3 also reports the performances of the subjects and their behavioral types based on two criteria: the relative frequencies of chosen numbers ('choice') and the frequency of changing one's choices from one round to another ('change'). For the choice criterion, subjects are considered to be MSE subjects if their choice frequencies were not statistically different, at 5% significance level, from those predicted by the MSE according to Kolmogorov-Smirnov tests. Otherwise, they are considered to be non-MSE subjects. For the change criterion, subjects are considered to be MSE subjects if their frequency of changing their choices between two consecutive rounds lies within the 95% confidence interval predicted by the MSE. Otherwise, they are considered to be non-MSE subjects.

As one can note from the table, while most of the subjects who are classified as non-MSE under the choice criterion are also classified as non-MSE under the change criterion, this is not the case for those who are classified as MSE under the choice criterion. Between 35 and 45% of subjects classified as MSE subjects by the choice criterion did not change their choices between two consecutive rounds as frequently as the MSE predicts.

[2] The English translation of the experiment's instructions is available upon request.

Table 3. Relative frequencies of observed winning numbers and summaries of the performances of the subjects

a. LUIG 33

	Expr.	MSE
Winning number		
0	12.88%	13.84%
1	40.50%	39.99%
2	26.88%	23.09%
3	19.75%	23.09%
Average #wins	14.52	
Average #changes	21.51	
Cor(#wins, #changes)	−0.436	
#Subjects		
MSE (choice)	29	
MSE (change)	0	
MSE (choice + change)	42	
Non-MSE	25	

b. LUIG 34

	Expr.	MSE
Winning number		
0	8.88%	11.80%
1	43.31%	40.38%
2	29.18%	22.20%
3	15.94%	12.81%
4	2.69%	12.81%
Average #wins	15.19	
Average #changes	21.02	
Cor(#wins, #changes)	−0.305	
#Subjects		
MSE (choice)	27	
MSE (change)	3	
MSE (choice + change)	33	
Non-MSE	33	

c. LUIG 43

	Expr.	MSE
Winning number		
0	33.33%	32.91%
1	27.50%	30.09%
2	32.17%	28.60%
3	7.00%	8.40%
Average #wins	8.33	
Average #changes	20.53	
Cor(#wins, #changes)	−0.320	
#Subjects		
MSE (choice)	32	
MSE (change)	4	
MSE (choice + change)	39	
Non-MSE	21	

d. LUIG 44

	Expr.	MSE
Winning number		
0	27.17%	32.63%
1	32.42%	30.17%
2	30.41%	28.63%
3	8.33%	8.47%
4	1.67%	0.01%
Average #wins	9.10	
Average #changes	21.18	
Cor(#wins, #changes)	−0.293	
#Subjects		
MSE (choice)	19	
MSE (change)	4	
MSE (choice + change)	37	
Non-MSE	36	

4 Computational Evolutionary Competition

In the laboratory experiment by Yamada and Hanaki, they observed that the number of wins for each subject was negatively correlated to that of changes in every game setup. That means, it may be fine to keep choosing a number to win LUIGs. But, it was not at that moment sure whether such sticking behaviors were really successful. Here, a computational experiment about evolutionary competition is employed to see the effectiveness of such behaviors. For this purpose, several typical learning models are employed and the parameters of the models are then estimated for the experiment.

4.1 Learning models

The learning models employed here are as follows:

– Adaptive learning (AL)
  An AL player i has a propensity $a_i^k(t)$ for number k $(k = 1, \cdots, M)$ at the beginning of round t. Before the start of a game, she is assumed to have the same non-negative propensities for all the possible integers, namely $a_i^j(0) = a_i^k(0) \geq 0$ for $j \neq k$.
  At every round, she chooses one integer according to the following exponential selection rule

$$p_i^k(t) = \frac{\exp(\lambda_a \cdot a_i^k(t))}{\sum_{k'=1}^{M} \exp(\lambda_a \cdot a_i^{k'}(t))} \tag{1}$$

  where $p_i^k(t)$ is i's selection probability for integer k at round t, and λ_a is a positive constant called sensitivity parameter ([1, 5]).
  After a round, propensities are updates as

$$a_i^k(t+1) = (1 - \phi_a)a_i^k(t) + 1_{\{k,s_i(t)\}}\psi_a R \tag{2}$$

  where ϕ_a and ψ_a are positive constants called learning parameter ([1, 5]), $1_{\{\cdot\}}$ is the indicator function that takes value 1 if $k = s_i(t)$, and 0 otherwise. Here $s_i(t)$ is the number that player i has actually chosen at round t, and R is the payoff received. Note that the model is called 'cumulative' if $\psi_a = 1$ and 'averaging' if $\psi_a = 1 - \phi_a$.
– Naive imitation (NI)
  Players using this model follow a winning number regardless of whether they are a winner or not. When "no-winner" situation happens, they choose the preceding number.
  While the selection rule is the same as that in AL model, the updating rule is expressed in the following:

$$a_i^k(t+1) = (1 - \phi_n)a_i^k(t) + 1_{\{k,v(t)\}}\psi_n R$$

  where $v(t)$ is a winning number at round t.
– Level-1 thinking
  Level-1 player always chooses 1.

Table 4. The number of subjects

	AL	NI	Level1	Level-2	Others
LUIG 33	58	35	1	2	0
LUIG 34	55	40	1	0	0
LUIG 43	56	38	1	1	0
LUIG 44	40	52	2	1	1

- Level-2 thinking
  Level-2 player always chooses 2.
- Other sticking behavior
  Players using this model always chooses only one number, but 3 or larger. Since level-k ($k > 0$) players cover one of the two numbers, 1 and 2, sticking to 3 or larger cannot be dealt with. Accordingly, such players are classified into this model.

To determine a learning model for every subject, the author assumes the following points: First, all initial propensities in Game 1 are set to zero, namely the subjects did not have any prior belief to others or view to the game. Second, the propensities of every subject at the beginning of Game 2 succeed to those at the end of Game 1. If the upper limit is different from the games, the corresponding propensities are not used ($M = 4$ in Game 1 $\rightarrow M = 3$ in Game 2) or set to zero ($M = 3$ in Game 1 $\rightarrow M = 4$ in Game 2). Then, the learning model with the best log likelihood is employed for the simulation[3]. Note that the subjects who did not change at all in a game belong to one of the three models, Level-1, Level-2 or other sticking behavior regardless the log likelihood. Table 4 summarizes the number of subjects for each learning model in each LUIG.

4.2 Setup

The author develops the following evolutionary competition algorithm: Every strategy i initially has the same existence fraction $w(i, 1) = 1/96$ for each LUIG. A generation has 5000 LUIGs each of which has 100 rounds. In each game, three or four learning models are randomly selected in accordance with the existence fraction. Therefore, one learning model is expected to play the games approximately 100 times or more.

For each learning model i, the average number of wins it earned is calculated, averaged over all simulations in a single generation. It is denoted as $P(i, g)$. Also, the average number of wins for all the learning models is calculated in the same way. It is denoted as $Q(g)$.

After each generation of 5000 LUIG games, the weights of the different learning models are updated on the basis of how well they played compared to the whole population of learning models. The updating rule is determined as follows:

$$\bar{w}(i, g+1) = \max\{(1 + \delta[P(i, g) - Q(g)])w(g, i), 0\}$$

[3] 'optim' function in R is used for estimation.

where δ is selection parameter taking a positive value. If a learning model performs better in a generation, its fraction increases. On the other hand, if it performers worse than average, such learning strategy becomes extinct, namely $\bar{w}(i, g+1) = 0$. The new fraction is set to

$$w(i, g+1) = \frac{\bar{w}(i, g+1)}{\sum_{i'} \bar{w}(i', g+1)} .$$

The experimental setup is similar to that in Linde et al. [10], namely, 10 simulation runs for each LUIG, each of which has 500 generations. The selection parameter δ is fixed at 0.05.

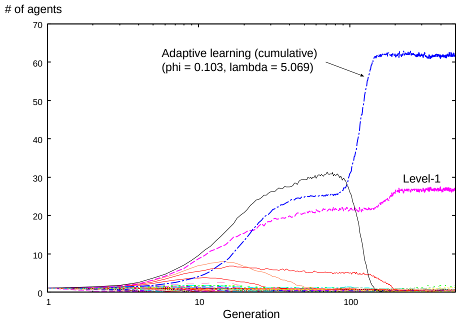
4.3 Results

Figure 1 shows what kind of estimated learning models survive for each LUIG. Each line means the average value over the 10 simulation runs. The learning models with 10 or more agents at the end of the simulations on average are given. Only a few estimated strategies eventually survived. In particular, there are always one or two Level-1 subject(s) in each LUIG. The next one is that adaptive learning models seem to perform better than naive imitation models.

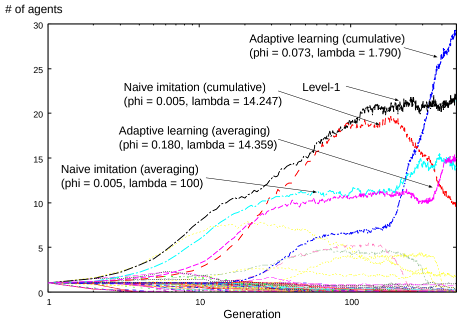
The point the author should see is whether the subjects with surviving learning models played well in the laboratory experiment. Indeed, p-values of Mann-Whitney test w.r.t. the number of wins and changes show that the number of changes for the surviving subjects is significantly less frequent than the others (0.016 for LUIG 33, 0.269 for LUIG 34, 0.338 for LUIG 43 and 0.024 for LUIG 44) meanwhile in some setup that does not always hold for the number of wins (0.020 for LUIG 33, 0.003 for LUIG 34, 0.007 for LUIG 43 and 0.004 for LUIG 44). To summarize, sticking behavior is more likely to win more and eventually survive in small-sized LUIGs.

Next, Figure 2 displays the time series plot of the average game efficiency (situations where there is a winner) and the average frequency of changes. The average number of changes steadily decreased over the generations in all LUIGs because only a few sticking behaviors were successful in this competition. On the other hand, there is an apparent difference between three-person LUIGs and four-person LUIGs; The game efficiency was slightly improved in three-person LUIGs meanwhile those were not in four-person LUIGs. Such a difference stems from the set of asymmetric pure strategy equilibria listed as follows:

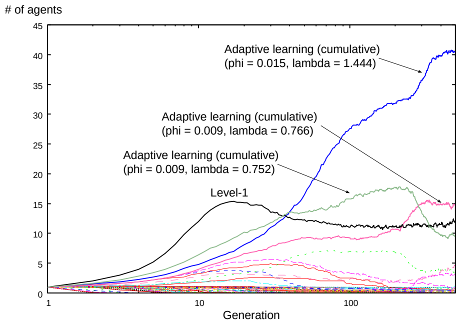
- Three-person LUIGs
  {1, 1, 2}, {1, 1, 3}, {1, 1, 4}, {1, 2, 2}, {1, 2, 3}, {1, 2, 4}, {1, 3, 3}, {1, 3, 4}, {1, 4, 4}
- Four-person LUIGs
  {1, 1, 1, 2}, {1, 1, 2, 2}, {1, 1, 2, 3}, {1, 1, 2, 4}, {1, 2, 2, 2}, {1, 2, 2, 3}, {1, 2, 2, 4}, {1, 2, 3, 3}, {1, 2, 3, 4}, {1, 2, 4, 4}, {1, 3, 3, 3}, {1, 3, 3, 4}, {1, 3, 4, 4}, {1, 4, 4, 4}



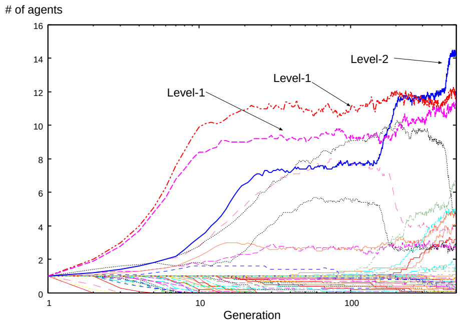
a. LUIG 33



b. LUIG 34



c. LUIG 43



d. LUIG 44

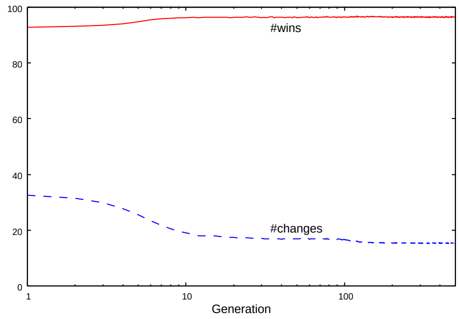
Fig. 1. Time series plot of the existence of agents for all agents over 10 simulation runs. The agents with 10 or more populations at the end of generation 500 are introduced with the estimated parameters.

Among these asymmetric pure strategy equilibria, the one, {1, 1, 2, 2}, is of particular importance because only this equilibrium does not have a winner. Hence, once this situation is reached, it is impossible to deviate, no-winner easily continues.

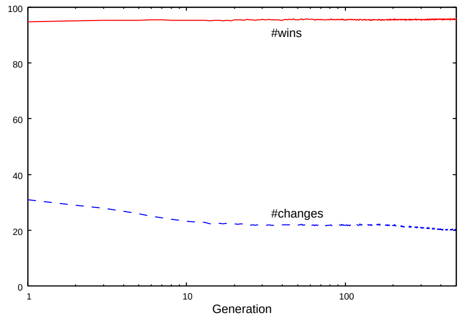
4.4 Discussion

LUIGs are sometimes considered as similar to MGs in that players are asked to behave differently from the others. The major differences are: (1) LUIGs have two or more options while MGs have only two, (2) Only one player can win in LUIGs while at most $(N-1)/2$ players can win in MGs, (3) The unique symmetric mixed strategy equilibrium depends on both the number of players and that of options in LUIGs. Yet, in MGs, choosing an option with equal probability, namely 0.5, always holds for MGs regardless of the number of players.

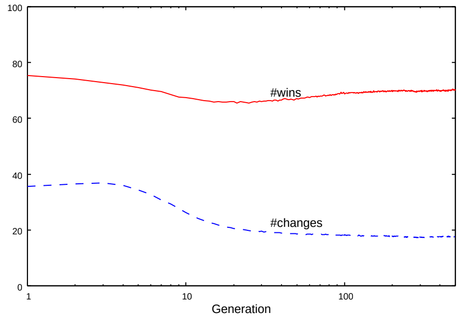
Having in mind the above similarities and differences, the author is going to see the results of evolutionary computation for MG of five by Linde et al. briefly and then discuss behavioral and constructional differences between the two games.



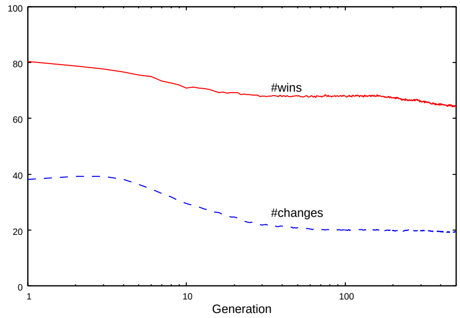
a. LUIG 33



b. LUIG 34



c. LUIG 43



d. LUIG 44

Fig. 2. Time series plot of average numbers of wins and changes for all agents over 10 simulation runs

Linde et al. hold a strategy experiment which consists of five rounds [10]. In each round, the subjects were asked to (improve and) submit their strategy for 100-period MG of five. The information for decision-making is whether s/he changed his/her option and the number of players with his/her option for each of the last five periods. Based on them, s/he writes a list of IF-THEN statements with probabilities to change his/her option. After computational experiments for all the possible combinations are pursued, the subjects receive the information about the average points earned for improving the strategies.

In their experiment, totally 42 subjects participated and the number of unique strategies classified was 107. Using the strategies, Linde et al. conducted another computational experiments, evolutionary computation. Their main results are: (i) Only four of 107 strategies survived. (ii) The surviving strategies rarely change, but occasionally do no to get stuck in losing situations. (iii) “Never change” strategy was one of the four surviving ones, but its performance is not so good as the other three. (iv) As the generations go on, the average frequency of changing options decreases and the game efficiency improves towards the full-mark, 40%.

Compared to the results in Section 3, two substantial differences between LUIGs and MGs can be addressed: First, sticking behavior performs well in

LUIGs, but not so in MGs. In other words, in MGs, taking into account the situations s/he faces is necessary to play well in MGs albeit the strategy does not need frequent changes. Second, cooperative behavior improves the game efficiency in MGs but pure competitive aspect of LUIGs does not yield such a situation. In particular, the game efficiency in four-person LUIGs deteriorated because of the asymmetric pure strategy equilibrium, $\{1, 1, 2, 2\}$.

5 Concluding Remark

This study examines what kind of behavioral models are successful in small-sized lowest unique integer games and discusses the differences between LUIGs and MGs by evolutionary competition approach. The behavioral models are obtained by estimating the parameters from the behavioral data in the laboratory experiment by Yamada and Hanaki [18]. Then computational experiment in line with the work by Linde et al. [10] is pursued. The main but preliminary findings are in the following: First, sticking behaviors such as Level-1 and Level-2 perform well both in evolutionary competition and in laboratory experiment. Second, as the generations goes by, the frequency of average changes decreases while the game efficiency differs from the game setup, namely improved in three-person LUIGs but not in four-person LUIGs. This stems from the structure of asymmetric pure strategy equilibria. Third, successful strategies in MGs take into account the situations to avoid getting stuck and cooperative aspect eventually improves the game efficiency. This is not computationally observed in LUIGs.

Since this study deals with the estimated learning models, unlike in Linde et al., there may be better models for some of the behavioral data in laboratory experiment. Hence, as done by Linde et al., it is necessary to conduct another laboratory experiment where subjects are asked to elicit their decisions to play LUIGs. Another future work includes larger-sized experiment to see whether similar behaviors and game dynamics are also observed. This comes form the empirical finding by Östling et al. [13].

Acknowledgment

The author thanks Nobuyuki Hanaki at GREDEG, Université de Nice Sophia Antipolis, Université Côte d'Azur, and IUF, and Angela Sutan and her colleagues at Laboratoire d'Expérimentation en Sciences Sociales et Analyse des Comportements (LESSAC) at Burgundy School of Business for their help in carrying out the experiments. The author also thanks Jean-Sébastien Gharbi for his help in preparing the experiment's instructions in French. Financial support from Japan Society for the Promotion of Science (JSPS) Grant-in-Aid for Young Scientists (B) (24710163) and Grant-in-Aid (c) (15K01180), from Canon Europe Foundation under a 2013 Research Fellowship Program, and from JSPS and ANR under the Joint Research Project, Japan – France CHORUS Program, "Behavioural and cognitive foundations for agent-based models (BECOA)" (ANR-11-FRJA-0002) is gratefully acknowledged.

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